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ANOTHER PRESENTATION FOR THE K_2 OF A LOCAL DOMAIN

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Introduction

It is well known that the K_2 of a field F has the following presentation as an Abelian group:

generators are symbols $\{a,b\}$, where $a,b\in F^*$;

defining relations are

$$\{b,a\}=\{a,b\}^{-1},\qquad \{a,b\}\{a,c\}=\{a,bc\},\qquad \{a,1-a\}=1.$$

This is the presentation given by Matsumoto [4], see also Milnor [5]. In this presentation the symbol $\{a,b\}$ corresponds to $h_{12}(ab)h_{12}(a)^{-1}h_{12}(b)^{-1}$.

I will prove in this paper that the K_2 of a local domain R has exactly the same presentation, provided we generalise the conditions $a,b\in F^*$ to the conditions

$$a\neq 0,\quad b\neq 0,\quad a\mid 1-b,\quad b\mid 1-a.$$

The element of $K_2(R)$ to which the symbol $\{a,b\}$ corresponds is

$$x_{21}\left(1-\frac{1-a}{b}\cdot\frac{1-b}{a}\right)x_{12}(-a)x_{21}\left(\frac{1-b}{a}\right)x_{12}\left(-\frac{1-a}{b}\right)x_{21}(b)w_{12}(1). \qquad (*)$$

The proof of this depends on the presentation for the K_2 of a local ring given by Van der Kallen, Maazen and Stienstra, see [2].

In Section 1 a symbol $\langle a,b,c\rangle_*$ is introduced, which is slightly more general (the ring may have zero divisors). This symbol is in Section 2 compared with the Dennis–Stein symbol $\langle a,b\rangle$ and the Steinberg symbol $\{a,b\}$. In Section 3 the symbol $\{a,b\}$ corresponding to $(*)$ above is considered. Finally, in Section 4 I will show that the symbols $\langle a,b,c\rangle_*$ generate the K_2 of a commutative ring satisfying the stable range condition SR_2.

1. The symbol $\langle a,b,c\rangle_*$

1.1. Let R be a commutative ring. For $a,b,c\in R$ satisfying

$$1-a-c+abc=0$$

we will define an element $\langle a,b,c\rangle_*\in K_2(n,R)$ $(n\geq 2)$. Its definition is motivated by the following lemma.

1.2. Lemma. *Let R be a commutative ring. For $a,b,c,d,e\in R$ the following conditions are equivalent:*

(i) $$\begin{pmatrix}1 & -a\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}\begin{pmatrix}1 & -c\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ d & 1\end{pmatrix}\begin{pmatrix}1 & -e\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix};$$

(ii) $$d=1-ab,\quad e=1-bc,\quad a=1-cd,\quad b=1-de,\quad c=1-ea;$$

(iii) $$1-a-c+abc=0,\quad d=1-ab,\quad e=1-bc.$$

1.3. The 5-tuple (a,b,c,d,e) is called a *cycle* in R if it satisfies one of the equivalent conditions in the lemma. Note the symmetry (condition (ii)), and also the fact that a cycle is determined by three consecutive elements of the cycle.

1.4. Definition. Put $V(R)=\{(a,b,c)\in R^3\mid 1-a-c+abc=0\}$. For $(a,b,c)\in V(R)$ and i, j such that $1\leq i,\ j\leq n,\ i\neq j$, define $\langle a,b,c\rangle_{ij}$ to be the following element of $\mathrm{St}(n,R)$ (where $n\geq 2$):

$$x_{ij}^{-a}x_{ji}^{b}x_{ij}^{-c}x_{ji}^{1-ab}x_{ij}^{-(1-bc)}w_{ij}(1).$$

Then from Lemma 1.2 it follows that $\langle a,b,c\rangle_{ij}\in K_2(n,R)$.

1.5. Theorem. *Let $n\geq 3$. We have the following identities in $K_2(n,R)$:*

(i) *$\langle a,b,c\rangle_{ij}$ is independent of ij (and will therefore be written as $\langle a,b,c\rangle_*$);*

(ii) $\langle a,b,c\rangle_*=\langle b,c,1-ab\rangle_*$;

(iii) $\langle c,b,a\rangle_*=\langle a,b,c\rangle_*^{-1}$;

(iv) $\langle a,bc,d\rangle_*\langle b,ca,e\rangle_*=\langle ab,c,d+ae\rangle_*$.

(Note that (ii) implies that $\langle a,b,c\rangle_*$ only depends on three consecutive elements of the cycle $(a,b,c,1-ab,1-bc)$, and that (iii) states that reversion of the order in the cycle means inversion in the group $K_2(n,R)$).

Proof. For (i) see [5, Section 9].

(ii) $\langle a,b,c\rangle_{12}=x_{12}^{-a}w_{12}(1)x_{12}^{-b}x_{21}^{c}x_{12}^{-(1-ab)}x_{21}^{1-bc}=\langle b,c,1-ab\rangle_{12}$.

(iii) $\langle c,b,a\rangle_{12}=w_{12}(1)x_{21}^{c}x_{12}^{-b}x_{21}^{a}x_{12}^{-(1-bc)}x_{21}^{1-ab}=\langle 1-ab,1-bc,a\rangle_{21}^{-1}=\langle a,b,c\rangle_{21}^{-1}$.

(iv) We have to prove

$$\langle a,bc,d\rangle_*\langle b,ca,e\rangle_*\langle c,ab,f\rangle_*=1$$

where $f=1-c(d+ae)$. An elementary calculation shows that also $d=1-a(e+bf)$ and $e=1-b(f+cd)$. We will use Philip Hall's identity

$$^{y}[x,[y^{-1},z]]\cdot {}^{z}[y,[z^{-1},x]]\cdot {}^{x}[z,[x^{-1},y]]=1$$

where $[x,y]$ stands for $xyx^{-1}y^{-1}$ and ^{y}x for yxy^{-1}. In this identity make the following substitutions:

$$x=x_\alpha^{-a},\qquad y=x_\beta^{-b},\qquad z=x_\gamma^{-c},$$

where $\alpha=12$, $\beta=23$ and $\gamma=31$. Also note that for $(a,b,c)\in V(R)$ we have the identity

$$[x_\alpha^{-a},x_{-\alpha}^{-b}]=\langle a,b,c\rangle_\alpha^{-1}x_\alpha^{-(a+c)}x_{-\alpha}^{1-ab}x_\alpha^{-b-(1-bc)}w_\alpha(1).$$

We obtain

$$\begin{aligned}
&\langle a,bc,d\rangle_*\langle b,ca,e\rangle_*\langle c,ab,f\rangle_*=\\
&=x_\beta^{-b}x_\alpha^{-(a+d)}x_{-\alpha}^{1-abc}x_\alpha^{-bc-(1-bcd)}w_\alpha(1)x_\beta^{b}\\
&\quad\cdot x_\gamma^{-c}x_\beta^{-(b+e)}x_{-\beta}^{1-abc}x_\beta^{-ca-(1-cae)}w_\beta(1)x_\gamma^{c}\\
&\quad\cdot x_\alpha^{-a}x_\gamma^{-(c+f)}x_{-\gamma}^{1-abc}x_\gamma^{-ab-(1-abf)}w_\gamma(1)x_\alpha^{a}\\
&=x_{-\gamma}^{-ab}x_\beta^{-b}x_\alpha^{-d}x_{-\alpha}^{1-abc}x_\alpha^{-bc-(1-bcd)}w_\alpha(1)\\
&\quad\cdot x_{-\alpha}^{-bc}x_\gamma^{-c}x_\beta^{-e}x_{-\beta}^{1-abc}x_\beta^{-ca-(1-cae)}w_\beta(1)\\
&\quad\cdot x_{-\beta}^{-ca}x_\alpha^{-a}x_\gamma^{-f}x_{-\gamma}^{1-abc}x_\gamma^{-ab-(1-abf)}w_\gamma(1)\\
&=x_\beta^{-b}x_\alpha^{-d}x_{-\alpha}^{1-abc}x_\alpha^{-(1-bcd)}w_\alpha(1)\\
&\quad\cdot x_\gamma^{-c}x_\beta^{-e}x_{-\beta}^{1-abc}x_\beta^{-(1-cae)}w_\beta(1)\\
&\quad\cdot x_\alpha^{-a}x_\gamma^{-f}x_{-\gamma}^{1-abc}x_\gamma^{-(1-abf)}w_\gamma(1)\\
&=x_{-\gamma}^{-bd}x_\alpha^{-d}x_\beta^{-b}x_{-\alpha}^{1-abc}x_\alpha^{-(1-bcd)}w_\alpha(1)\\
&\quad\cdot x_{-\alpha}^{-ce}x_\beta^{-e}x_\gamma^{-c}x_{-\beta}^{1-abc}x_\beta^{-(1-cae)}w_\beta(1)\\
&\quad\cdot x_{-\beta}^{-af}x_\gamma^{-f}x_\alpha^{-a}x_{-\gamma}^{1-abc}x_\gamma^{-(1-abf)}w_\gamma(1)\\
&=x_\alpha^{-d}x_\beta^{-b}x_{-\alpha}^{1-abc}x_\alpha^{-f}w_\alpha(1)\\
&\quad\cdot x_\beta^{-e}x_\gamma^{-c}x_{-\beta}^{1-abc}x_\beta^{-d}w_\beta(1)\\
&\quad\cdot x_\gamma^{-f}x_\alpha^{-a}x_{-\gamma}^{1-abc}x_\gamma^{-e}w_\gamma(1)\\
&=x_\beta^{-1}x_{-\gamma}^{-e}x_{-\alpha}^{1-abc}x_\alpha^{-f}w_\alpha(1)\\
&\quad\cdot x_\gamma^{-1}x_{-\alpha}^{-f}x_{-\beta}^{1-abc}x_\beta^{-d}w_\beta(1)\\
&\quad\cdot x_\alpha^{-1}x_{-\beta}^{-d}x_{-\gamma}^{1-abc}x_\gamma^{-e}w_\gamma(1)\\
&=x_\beta^{-1}x_{-\alpha}^{1-abc}w_\alpha(1)x_\gamma^{-1}x_{-\beta}^{1-abc}w_\beta(1)x_\alpha^{-1}x_{-\gamma}^{1-abc}w_\gamma(1)
\end{aligned}$$

$$= x_\beta^{-1} x_{-\alpha}^{1-abc} x_{-\beta}^{1} x_\gamma^{1-abc} x_\beta^{-1} x_{-\alpha}^{-(1-abc)} w_\alpha(1) w_\beta(1) w_\gamma(1)$$
$$= x_\beta^{-1} x_{-\beta}^{1} x_{-\alpha}^{1-abc} x_\beta^{-1} x_{-\alpha}^{-(1-abc)} w_\alpha(1) w_\beta(1) w_\gamma(1)$$
$$= w_\beta(-1) w_\alpha(1) w_\beta(1) w_\gamma(1) = w_{-\gamma}(1) w_\gamma(1) = w_\gamma(-1) w_\gamma(1) = 1.$$

1.6. Definition. Let R be a commutative ring. The Abelian group $C(R)$ is defined by generators and relations as follows:

generators are symbols $\langle a,b,c\rangle$, where $(a,b,c)\in V(R)$;

defining relations are

(C1) $$\langle a,b,c\rangle = \langle b,c,1-ab\rangle,$$

(C2) $$\langle c,b,a\rangle = \langle a,b,c\rangle^{-1},$$

(C3) $$\langle a,bc,d\rangle\langle b,ca,e\rangle = \langle ab,c,d+ae\rangle.$$

1.7. From Theorem 1.5 it follows that for $n\geq 3$ a homomorphism

$$\phi : C(R)\to K_2(n,R)$$

is defined by $\phi(\langle a,b,c\rangle) = \langle a,b,c\rangle_*$.

1.8. Proposition. (i) *If one of the five elements in the cycle $(a,b,c,1-ab,1-bc)$ in R is zero, then $\langle a,b,c\rangle = 1$.*

(ii). *Let $a,b,c,d\in R$ be such that $1-a=d(1-ab)(1-ac)$, then*

$$\langle a,b,d(1-ac)\rangle\langle a,c,d(1-ab)\rangle = \langle a,b+c-abc,d\rangle.$$

Proof. From (C1) it follows that, for the proof of (i), we may assume $b=0$. In (C3) substitute for a, b, c, d, e respectively a, 0, 0, $1-a$, 1. Then (C3) becomes

$$\langle a,0,1-a\rangle\langle 0,0,1\rangle = \langle 0,0,1\rangle.$$

Hence $\langle a,0,1-a\rangle = 1$.

Proof of (ii):

$$\langle a,b,d(1-ac)\rangle\langle a,c,d(1-ab)\rangle =$$
$$= \langle b,d(1-ac),1-ab\rangle\langle c,d(1-ab),1-ac\rangle \qquad \text{(C1)}$$
$$= (\langle 1-ab,d(1-ac),b\rangle\langle 1-ac,d(1-ab),c\rangle)^{-1} \qquad \text{(C2)}$$
$$= \langle (1-ab)(1-ac),d,b+(1-ab)c\rangle^{-1} \qquad \text{(C3)}$$
$$= \langle d,b+c-abc,a\rangle^{-1} \qquad \text{(C1)}$$
$$= \langle a,b+c-abc,d\rangle \qquad \text{(C2).}$$

2. Comparison with Dennis–Stein symbols and Steinberg symbols

2.1. Let $a, b \in R$ such that $1-ab \in R^*$. The Dennis–Stein symbol $\langle a,b\rangle_*$ is defined as follows:

$$\langle a,b\rangle_* = x_{21}^{-b(1-ab)^{-1}} x_{12}^{-a} x_{21}^{b} x_{12}^{a(1-ab)^{-1}} h_{12}(1-ab)^{-1}.$$

This symbol was introduced by Dennis and Stein [6]. Notice however the notational difference: $\langle a,b\rangle_*$ corresponds to $\langle -a,b\rangle_*$ in [6]. If $1-ab\in R^*$, then $(a,b,(1-a)(1-ab)^{-1}) \in V(R)$. An easy computation shows that

$$\langle a,b\rangle_* = \langle a,b,(1-a)(1-ab)^{-1}\rangle_*.$$

2.2. Let the Abelian group $D(R)$ be defined by generators and relations as follows:
generators are $\langle a,b\rangle$, where $a,b\in R$, $1-ab\in R^*$;
defining relations are

$$\langle b,a\rangle = \langle a,b\rangle^{-1}, \qquad \langle a,b\rangle\langle a,c\rangle = \langle a,b+c-abc\rangle,$$

$$\langle ab,c\rangle = \langle a,bc\rangle\langle b,ca\rangle.$$

2.3. Proposition. *There is a homomorphism $\psi: D(R)\to C(R)$ such that $\psi(\langle a,b\rangle) = \langle a,b,(1-a)(1-ab)^{-1}\rangle$. (The map $\phi\psi: D(R)\to K_2(n,R)$, $n\geq 3$, sends $\langle a,b\rangle$ to $\langle a,b\rangle_*$).*

Proof. This follows from (C2), Proposition (1.8)(ii), and (C3).

2.4. In general the image of $C(R)$ in $K_2(R)$ is larger than the image of $D(R)$ in $K_2(R)$. To see this take the universal example:

$$R = \mathbb{Z}[X,Y,Z]/(1-X-Z+XYZ).$$

It is an easy exercise to show that $R\to\mathbb{C}\otimes_{\mathbb{Z}} R$ is injective and that $(\mathbb{C}\otimes_{\mathbb{Z}} R)^* = \mathbb{C}^*$, and hence also $R^*=\mathbb{Z}^*$. From this it easily follows that $\mathbb{Z}\to R$ induces an isomorphism $D(\mathbb{Z}) \xrightarrow{\simeq} D(R)$.

Let S be the ring $\mathbb{Z}[X,Z]/(XZ)$, and $f: R\to S$ the surjective ring homomorphism given by $\bar{X}\mapsto 1-\bar{X}$, $\bar{Y}\mapsto 1$, $\bar{Z}\mapsto 1-\bar{Z}$. Consider the homomorphisms

$$D(R) \xrightarrow{\psi} C(R) \xrightarrow{\phi} K_2(R) \xrightarrow{f_*} K_2(S).$$

We have $f_*\phi\langle \bar{X},\bar{Y},\bar{Z}\rangle = f_*\langle \bar{X},\bar{Y},\bar{Z}\rangle_* = \langle 1-\bar{X},1,1-\bar{Z}\rangle_* = \langle \bar{X},\bar{Z}\rangle_*$. Moreover, $K_2(S)\cong K_2(\mathbb{Z})\oplus\mathbb{Z}$, where the last summand is generated by $\langle \bar{X},\bar{Z}\rangle_*$; see [1], or [3, (4.6)].

Hence $\langle \bar{X},\bar{Y},\bar{Z}\rangle_* \in K_2(R)$ is of infinite order, and therefore not in the image of $D(R)$, since $D(R)\cong D(\mathbb{Z})$ which is a group or order 2.

2.5. An example of lower Krull dimension is

$$R=k[X,Z]/(1-X-Z+X^2Z)$$

where k is a field of characteristic $\neq 2$. Let S be the ring $k[X,X^{-1}]$, and $f:R\to S$ the surjective ring homomorphism given by $f(\bar{X})=X^{-1}-1$, $f(\bar{Z})=X$. In this case we have

$$D(k)\xrightarrow{\sim} D(R)\xrightarrow{\psi} C(R)\xrightarrow{\phi} K_2(R)\xrightarrow{f_*} K_2(S)\xrightarrow{\sim} K_2(k)\oplus k^*$$

and the composition maps $D(k)$ isomorphically onto the first summand $K_2(k)$. However, $\langle\bar{X},\bar{X},\bar{Z}\rangle\in C(R)$ maps to $-1\in k^*: f_*\phi\langle\bar{X},\bar{X},\bar{Z}\rangle=f_*\langle\bar{X},\bar{X},\bar{Z}\rangle_*=\langle X^{-1}-1,X^{-1}-1,X\rangle_*=\langle X,X,X\rangle_*=\{X,X\}_*=\{X,-1\}_*$.

2.6. Let $a,b\in R^*$. The Steinberg symbol $\{a,b\}_*$ is defined as follows:

$$\{a,b\}_*=h_{12}(ab)h_{12}(a)^{-1}h_{12}(b)^{-1}.$$

It is related to the Dennis–Stein symbol by

$$\{a,b\}_*=\langle a,(1-b)a^{-1}\rangle_*.$$

In fact, there is a homomorphism

$$S(R)\to D(R),\qquad \{a,b\}\to\langle a,(1-b)a^{-1}\rangle$$

where $S(R)$ is the Abelian group defined by generators and relations as follows:
generators are $\{a,b\}$, where $a,b\in R^*$;
defining relations are

$$\{b,a\}=\{a,b\}^{-1},\qquad \{a,b\}\{a,c\}=\{a,bc\},\qquad \{a,1-a\}=1.$$

Then the composite

$$S(R)\to D(R)\xrightarrow{\psi} C(R)\xrightarrow{\phi} K_2(R)$$

is the map $\{a,b\}\to\{a,b\}_*$. The image of $\{a,b\}$ in $C(R)$ is

$$\langle b,1-(1-a)(1-b)a^{-1}b^{-1},a\rangle.$$

2.7. Van der Kallen, Maazen and Stienstra proved that the map

$$D(R)\to K_2(n,R),\qquad \langle a,b\rangle\to\langle a,b\rangle_*$$

is an isomorphism for R a local ring and $n\geq 3$, see [2]. This map is the composite map

$$D(R)\xrightarrow{\psi} C(R)\xrightarrow{\phi} K_2(n,R).$$

So $\phi:C(R)\to K_2(n,R)$ is also an isomorphism, because ψ is surjective: if $(a,b,c)\in V(R)$, then the elements a, b, c cannot all lie in the maximal ideal of R, say $a\in R^*$, but then

$$\langle a,b,c\rangle=\langle c,1-ab,1-bc\rangle=\psi\langle c,1-ab\rangle.$$

3. Integral domains

3.1. Let R be an integral domain. In this case a generator $\langle a,b,c\rangle \in C(R)$, where $(a,b,c) \in V(R)$, is completely determined by a and c:
if $ac=0$, then $\langle a,b,c\rangle = 1$ (see (1.8)).
if $ac\neq 0$, then

$$b = 1 - \frac{1-a}{c}\cdot\frac{1-c}{a}.$$

In the last case the cycle determined by a, b, c is

$$\left(a, 1-\frac{1-a}{c}\cdot\frac{1-c}{a}, c, \frac{1-a}{c}, \frac{1-c}{a}\right).$$

Therefore in this case: $(a,b,c)\in V(R)$ if and only if $a\,|\,c-1$ and $c\,|\,a-1$.

For $C(R)$ we will find a presentation in terms of the generators $\langle a,b,c\rangle$ where $ac\neq 0$. In view of 2.6 it is reasonable to denote them as $\{c,a\}$.

3.2. Definition. Let R be an integral domain. The Abelian group $C'(R)$ is defined in terms of generators and relations as follows:
generators are $\{c,a\}$, where $a,c\in R$, $ac\neq 0$, $1\,|\,1-c$, $c\,|\,1-a$;
defining relations are

(C′1) $\qquad \{a,c\} = \{c,a\}^{-1}$,

(C′2) $\qquad \{c,a\}\{c,b\} = \{c,ab\}$,

(C′3) $\qquad \{c,1-c\} = 1$

(whenever everything defined).

3.3. Proposition. *Let R be an integral domain. Then there is an isomorphism*

$$\chi : C'(R) \to C(R)$$

determined by

$$\chi\{c,a\} = \left\langle a, 1-\frac{1-c}{a}\cdot\frac{1-a}{c}, c\right\rangle.$$

Proof. Put $W(R) = \{(c,a)\in R\times R\,|\,ac\neq 0, a\,|\,1-c, c\,|\,1-a\}$. The map $\chi : W(R)\to C(R)$ induces a homomorphism $\chi : C'(R)\to C(R)$, as in the statement of the proposition, precisely when
(i) $\chi(a,c) = \chi(c,a)^{-1}$,
(ii) $\chi(c,a)\chi(c,b) = \chi(c,ab)$,
(iii) $\chi(c,1-c) = 1$.
(i) follows from (C2), and (iii) from Proposition (1.8)(i).

Proof of (ii):

$$\left\langle a, 1-\frac{1-a}{c}\cdot\frac{1-c}{a}, c\right\rangle \underset{(\mathrm{C1})}{\equiv} \left\langle \frac{1-a}{c}, \frac{1-c}{a}, a\right\rangle \underset{(\mathrm{C2})}{\equiv} \left\langle a, \frac{1-c}{a}, \frac{1-a}{c}\right\rangle^{-1}.$$

Hence

$$\begin{aligned}\chi(c,a)\chi(c,b) &= \left\langle a, \frac{1-c}{a}, \frac{1-a}{c}\right\rangle^{-1} \left\langle b, \frac{1-c}{b}, \frac{1-b}{c}\right\rangle^{-1} \\ &= \left\langle a, b\frac{1-c}{ab}, \frac{1-a}{c}\right\rangle^{-1} \left\langle b, a\frac{1-c}{ab}, \frac{1-b}{c}\right\rangle^{-1} \\ &\underset{(\mathrm{C3})}{\equiv} \left\langle ab, \frac{1-c}{ab}, \frac{1-ab}{c}\right\rangle^{-1} = \chi(c, ab).\end{aligned}$$

In order to prove that χ is an isomorphism we will show that it has an inverse. This amounts of showing that the map $\varrho: V(R)\to C'(R)$ defined by

$$\varrho(a,b,c) = \begin{cases} 1 & \text{when } ac = 0, \\ \{c,a\} & \text{when } ac \neq 0, \end{cases}$$

induces a homomorphism from $C(R)$ to $C'(R)$. We have to show:

(i) $\varrho(a,b,c) = \varrho(b,c,1-ab)$,

(ii) $\varrho(c,b,a) = \varrho(a,b,c)^{-1}$,

(iii) $\varrho(a,bc,d)\varrho(b,ca,e) = \varrho(ab,c,d+ae)$.

We will use the following lemma's:

Lemma 1. $\{c,a\} = \{a,(1-a)/c\}$ *if* $a \neq 1$.

Proof.

$$\{c,a\} \underset{(\mathrm{C'3})}{\equiv} \{1-a,a\}^{-1}\{c,a\} \underset{(\mathrm{C'2})}{\equiv} \left\{\frac{1-a}{c}, a\right\}^{-1} \underset{(\mathrm{C'1})}{\equiv} \left\{a, \frac{1-a}{c}\right\}.$$

Lemma 2. $\varrho(a,b,c) = 1$ *when one of the five elements of the cycle determined by* (a,b,c) *is equal to* 0 (*or equal to* 1, *which amounts to the same*).

Proof. $\varrho(a,b,c) = 1$ if $a = 0$ or $c = 0$ by definition of ϱ. If $b = 0$ then $a = 1-c$, and so $\varrho(a,b,c) = 1$ either by definition or by (C′3). If $1-ab = 0$, then $a = 1$, and so $\varrho(a,b,c) = 1$ either by definition or by (C′2). Similarly for $1-bc = 0$.

Proof of (i), (ii) and (iii): (ii) holds either by definition of ϱ or else by (C1); from Lemma 2 it follows that for the proof of (i) we may assume that $a \neq 0$, $c \neq 0$, $a \neq 1$, $c \neq 1-a$, but then by Lemma 1:

$$\begin{aligned}\varrho(a,b,c) = \{c,a\} = \left\{a, \frac{1-a}{c}\right\} = \left\{\frac{1-a}{c}, \frac{1-\frac{1-a}{c}}{a}\right\} \\ = \{1-ab, b\} = \varrho(b,c,1-ab).\end{aligned}$$

To prove (iii) we first apply (i) several times:

$$\varrho(a, bc, d) = \varrho(1-abc, 1-bcd, a),$$

$$\varrho(b, ca, e) = \varrho(1-abc, 1-cae, b),$$

$$\varrho(ab, c, d+ae) = \varrho(1-abc, 1-cd-cae, ab).$$

When $ab=0$ or $1-abc=0$, (iii) follows from Lemma 2. Otherwise it follows from (C′2).

3.4. Theorem. *Let R be a local domain. Then the map*

$$C'(R) \to K_2(n, R), \qquad \{c, a\} \mapsto \left\langle a, 1-\frac{1-a}{c}\cdot\frac{1-c}{a}, c\right\rangle_*$$

is an isomorphism for all $n \geq 3$.

Proof. This follows from Proposition 3.3 and 2.7.

4. The K_2 of a commutative ring satisfying SR_2 is generated by symbols $\langle a, b, c\rangle_*$

4.1. A ring R is said to satisfy the stable range condition SR_2 if for all pairs $(a, b) \in R^2$ such that $Ra + Rb = R$ there exists a $c \in R$ such that

$$a + bc \in R^*.$$

4.2. Theorem. *Let R be a commutative ring satisfying* SR_2. *Then the homomorphism*

$$\phi : C(R) \to K_2(n, R)$$

is surjective for all $n \geq 3$.

Proof. We will show that for $n \geq 3$

$$\mathrm{St}(n, R) = LULHC,$$

where: L is the subgroup generated by all x_{ij}^a with $i>j$, $a \in R$,
U is the subgroup generated by all x_{ij}^a with $i<j$, $a \in R$,
H is the subgroup generated by all $h_{ij}(u)$ with $u \in R^*$,
C is the subgroup generated by all $\langle a, b, c\rangle_*$ with $(a, b, c) \in V(R)$.
Since elements of type $x_{i+1,i}^a$ and $w_{i+1,i}(1)$ generate $\mathrm{St}(n, R)$, it suffices to show that $LULHC$ is invariant under left multiplication by $x_{i+1,i}^a$ and $w_{i+1,i}(1)$. For $x_{i+1,i}^a$ this is trivial.

For $w_{i+1,i}(1)$ we will need the following formula ($\alpha = i+1, i$)

$$w_\alpha(1) x_\alpha^{-a} x_{-\alpha}^{b} = x_\alpha^{u^{-1}(1-bc)} x_{-\alpha}^{-u(1-ab)} x_\alpha^{cu^{-2}} h_\alpha(u)^{-1} \langle au^{-1}, bu, cu^{-1}\rangle_* \qquad (*)$$

where $c, u \in R$ are such that

$$a + c(1 - ab) = u \in R^*.$$

Since clearly $Ra + R(1 - ab) = R$ it follows from SR_2 that such a $c \in R$ exists. The formula (*) is easily proved by conjugating

$$\langle au^{-1}, bu, cu^{-1}\rangle_* = x_\alpha^{-au^{-1}} x_{-\alpha}^{bu} x_\alpha^{-cu^{-1}} x_{-\alpha}^{1-ab} x_\alpha^{-(1-bc)} w_\alpha(1)$$

with $w_{i+1,j}(u)$, where $i \neq j \neq i+1$.

We have to prove that

$$w_\alpha(1) t_1 t_2 t_3 hk \in LULHC$$

for $t_1, t_2 \in L$, $t_2 \in U$, $h \in H$, $k \in C$.

Put $t_1 = t'_1 x_\alpha^{-a}$, $t_2 = x_{-\alpha}^b t'_2$, where t'_1 is a product of elements x_{lj}^r with $r \in R$, $l > j$, $(l,j) \neq \alpha$, and similarly t'_2 a product of elements x_{lj}^r with $r \in R$, $l < j$, $(l,j) \neq -\alpha$.

From formula (*) it follows that there exist $d, e, f \in R$, $h' \in H$, $k' \in C$ such that

$$w_\alpha(1) x_\alpha^{-a} x_{-\alpha}^b = x_\alpha^d x_{-\alpha}^e x_\alpha^f h' k'.$$

Now,

$$\begin{aligned} w_\alpha(1) t_1 t_2 t_3 hk &= w_\alpha(1) t'_1 x_\alpha^{-a} x_{-\alpha}^b t'_2 t_3 hk \\ &= t''_1 w_\alpha(1) x_\alpha^{-a} x_{-\alpha}^b t'_2 t_3 hk \qquad &&(\text{where } t''_1 \in L) \\ &= t''_1 x_\alpha^d x_{-\alpha}^e x_\alpha^f h' k' t'_2 t_3 hk \\ &= t''_1 x_\alpha^d x_{-\alpha}^e x_\alpha^f t''_2 t'_3 h'' k'' \qquad &&(\text{where } t''_2 \in U, t'_3 \in L, \\ & &&\ h'' \in H, k'' \in C) \\ &= t''_1 x_\alpha^d x_{-\alpha}^e t'''_2 x_\alpha^f t'_3 h'' k'' \qquad &&(\text{where } t'''_2 \in U) \\ &\in LULHC. \end{aligned}$$

If $x = t_1 t_2 t_3 hk \in K_2(n, R)$ with $t_1, t_3 \in L$, $t_2 \in U$, $h \in H$, $k \in C$, then it is easily seen that $t_1 t_2 t_3 = 1$.

So $x = hk$ with $h \in H \cap K_2(n, R)$ and $k \in C$. Since $H \cap K_2(n, R)$ is generated by elements of the form $\{r, s\}_*$, $r, s \in R^*$, we have $K_2(n, R) \subset C$.

4.3. Conjecture. For R a commutative ring satisfying SR_2 the homomorphism $\phi : C(R) \to K_2(n, R)$, for $n \geq 3$, actually is an isomorphism.

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